2) Theorems: Energy conservation and reciprocity

1. Poynting theorem

For bianisotropic medium, in f domain:

$$\begin{cases} \vec{\nabla}\times\vec{E} = -j\omega\vec{B} - \vec{J}_m \\ \vec{\nabla}\times\vec{H} = j\omega\vec{D} + \vec{J} \end{cases} \qquad \begin{cases} \vec{D} = \hat{\varepsilon}\vec{E} + \hat{\alpha}\vec{H} \\ \vec{B} = \hat{\mu}\vec{H} + \hat{\beta}\vec{E} \end{cases}$$

$\vec{J} = \vec{J}_c + \vec{J}_i$, $\vec{J}_c = \hat{\sigma}\vec{E}$ $\quad$ $\vec{J}_c$ conduction current, $\vec{J}_i$ external current

average Poynting vector = $Re(\vec{S})$

$$\Rightarrow \frac{1}{2}\vec{\nabla}\cdot(\vec{E}\times\vec{H}^*) = \vec{\nabla}\cdot\vec{S} = \frac{1}{2}\vec{H}^*\cdot\vec{\nabla}\times\vec{E} - \frac{1}{2}\vec{E}\cdot\vec{\nabla}\times\vec{H}^*$$

$$= -\frac{j\omega}{2}\left[\vec{H}^*\cdot\vec{B} - \vec{E}\cdot\vec{D}^*\right] - \frac{1}{2}\vec{E}\cdot\vec{J}_i^* - \frac{1}{2}\vec{E}\cdot\hat{\sigma}^*\cdot\vec{E}^* - \frac{1}{2}\vec{H}^*\cdot\vec{J}_m$$

Integrating over a volume V and interpretating the meaning of the real parts:

$$\underbrace{\oint \vec{S}\cdot d\vec{S}}_{\text{power oozing out of V through S}} = \int_V \Big[\underbrace{-\frac{j\omega}{2}(\vec{H}^*\cdot\vec{B} - \vec{E}\cdot\vec{D}^*)}_{\text{rate of change of the em energy stored in V}} \underbrace{-\frac{1}{2}\vec{E}\cdot\hat{\sigma}^*\vec{E}^*}_{\text{work given to the moving charges (currents)}} \underbrace{-\frac{1}{2}(\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{J}_m)}_{\text{power delivered by the sources}} \Big] dV$$

2. Passivity constraint

$\boxed{\text{Isotropic case}}$ $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E}$

$$\vec{\nabla} \times \vec{H} = j\omega \varepsilon \vec{E} + \sigma \vec{E} + \vec{J}_i = j\omega \left(\varepsilon + \frac{\sigma}{j\omega}\right) \vec{E} + \vec{J}_i = j\omega \underbrace{\varepsilon \left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}_{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}} \vec{E}$$

⇒ A conductivity is equivalent to an imaginary part of ε → no need to distinguish the Joule loss from the dielectric losses when ε, μ are complex

⇒ we now drop the σ term, keep only $\varepsilon = \varepsilon_R + j\varepsilon_I$, $\mu = \mu_R + j\mu_I$

Poynting theorem in a closed volume without sources:

$$\oint_S \vec{S} . dS = \int_V dV \left[-j\omega \frac{1}{2} \left(\vec{H}^* . \mu \vec{H} - \vec{E} . \varepsilon^* \vec{E}^* \right) \right] = \int_V dV \left[- \frac{j\omega}{2} \left(\mu_R |\vec{H}|^2 - \varepsilon_R |\vec{E}|^2 \right) + \frac{\omega}{2} \left(\mu_I |\vec{H}|^2 + \varepsilon_I |\vec{E}|^2 \right) \right]$$

reactive   active

$$\Rightarrow \mathrm{Re}\left[\oint_S \vec{S} . d\vec{S} \right] = \int_V \frac{\omega}{2} \left(\mu_I |\vec{H}|^2 + \varepsilon_I |\vec{E}|^2 \right) dV \leq 0 \quad \forall V, \forall (\vec{E}, \vec{H})$$

$$\boxed{\begin{array}{l} \text{passivity} \Rightarrow \mu_I, \varepsilon_I \leq 0 \\ \text{lossless} \Rightarrow \mu_I = \varepsilon_I = 0 \end{array}}$$

passivity constraints in isotropic media

$\boxed{\text{Bi-anisotropic case}}$ $\quad \vec{B} = \hat{\mu}\vec{H} + \hat{\beta}\vec{E} \quad , \quad \vec{D} = \hat{\varepsilon}\vec{E} + \hat{\alpha}\vec{H}$

$$\oint_S \vec{S}.\vec{dS} = \underbrace{\int -\frac{j\omega}{2}(\vec{H}^*.\vec{B} - \vec{E}.\vec{D}^*)dV}_{\substack{\text{Re} \Rightarrow \text{losses, absorption} \\ \text{Im} \Rightarrow \text{stored, reactive}}}$$

To be lossless, we need $\text{Re}\left\{-\frac{j\omega}{2}(\vec{H}^*.\vec{B} - \vec{E}.\vec{D}^*)\right\} = \text{Im}\left\{\vec{H}^*.\vec{B} - \vec{E}.\vec{D}^*\right\} = 0$

$$\Rightarrow \text{Im}\left\{\vec{H}^*.\hat{\mu}\vec{H} + \vec{H}^*.\hat{\beta}\vec{E} - \vec{E}.\hat{\varepsilon}^*\vec{E}^* - \vec{E}.\hat{\alpha}^*\vec{H}^*\right\} = 0$$

$$\text{Im}\, z = \frac{1}{2j}(z - z^*)$$

$$\Rightarrow \vec{H}^*.\hat{\mu}\vec{H} - \vec{H}.\hat{\mu}^*\vec{H}^* + \vec{H}^*.\hat{\beta}\vec{E} - \vec{H}.\hat{\beta}^*\vec{E}^* - \vec{E}.\hat{\varepsilon}^*\vec{E}^* + \vec{E}^*.\hat{\varepsilon}\vec{E} - \vec{E}.\hat{\alpha}^*\vec{H}^* + \vec{E}^*.\hat{\alpha}\vec{H} = 0$$

$$\vec{A}^*.\hat{m}\vec{B} = A_i^* m_{ij} B_j = B_j m_{ij} A_i^* = B_i m_{ji} A_j^* = \vec{B}.m^T\vec{A}^*$$

$$\Rightarrow \vec{H}.\underbrace{(\hat{\mu}^T - \hat{\mu}^*)}_{=0}\vec{H}^* + \vec{E}.\underbrace{(\hat{\varepsilon}^T - \hat{\varepsilon}^*)}_{=0}\vec{E}^* + \vec{E}.\underbrace{(\hat{\beta}^T - \alpha^*)}_{=0}\vec{H}^* + \vec{H}.\underbrace{(\hat{\alpha}^T - \hat{\beta}^*)}_{=0}\vec{E}^* = 0 \quad [\text{true } \forall(\vec{E},\vec{H})]$$

$$\Rightarrow \begin{cases} \mu^T = \mu^* \\ \varepsilon^T = \varepsilon^* \\ \beta^T = \alpha^* \end{cases} \Rightarrow \boxed{\begin{matrix} \mu = \mu^\dagger \\ \varepsilon = \varepsilon^\dagger \\ \beta = \alpha^\dagger \end{matrix}}$$

lossless conditions for bianisotropic media

$\boxed{\text{Chiral case}}$ (aka bi isotropic)

$$\hat{\mu} = \mu \quad , \quad \hat{\varepsilon} = \varepsilon \quad \Rightarrow \boxed{\mu \in \mathbb{R}, \varepsilon \in \mathbb{R}, \beta = \alpha^*} \Rightarrow \boxed{\begin{matrix} \vec{D} = \varepsilon\vec{E} + j\gamma\vec{H} \\ \vec{B} = \mu\vec{H} - j\gamma^*\vec{H} \end{matrix}}$$

lossless chiral

Side note on spatial dispersion

$$\begin{cases} \vec{D} = \varepsilon \vec{E} + j\gamma \vec{H} \\ \vec{B} = \mu \vec{H} - j\gamma^* \vec{E} \end{cases}$$ is a local constitutive relation. Yet, it is called

WEAK spatial dispersion

Why? Because $$\begin{cases} \vec{D} = \varepsilon \vec{E} + j\frac{\gamma}{\mu} \vec{B} + \frac{j\gamma}{\mu} j\gamma^* \vec{E} \\ \vec{B} = \vec{\nabla} \times \vec{E} \end{cases}$$

$$\Rightarrow \vec{D} = \left(\varepsilon - \frac{|\gamma|^2}{\mu}\right) \vec{E} - \frac{\gamma}{\omega\mu} (\vec{\nabla} \times \vec{E}) = f(\vec{E})$$

↳ derivative = nonlocal

In optics, μ is not well defined. [Landau Lifshitz book, Electrodynamics of CM]

We must keep $\mu = \mu_0$ and interpret magnetic-like effects as dielectric spatial dispersion:

$$\begin{cases} \vec{\nabla} \times \vec{E} = -j\omega \vec{B} = -j\omega\mu \vec{H} \\ \vec{\nabla} \times \vec{H} = j\omega \varepsilon \vec{E} + \vec{J} \end{cases}$$ ← let's add $\frac{\mu - \mu_0}{\mu_0} \vec{\nabla} \times \vec{H}$ on both sides

$$\Rightarrow \vec{\nabla} \times \underbrace{\left(\vec{H} + \frac{\mu - \mu_0}{\mu_0} \vec{H}\right)}_{\vec{H}_{eq}} = j\omega \underbrace{\left(\varepsilon \vec{E} + \frac{\mu - \mu_0}{j\omega\mu_0} \vec{\nabla} \times \vec{H}\right)}_{\vec{D}_{eq}} + \vec{J}$$

← redefine what we call $\vec{H}$ and $\vec{D}$

$$\Rightarrow \begin{cases} \vec{\nabla} \times \vec{E} = -j\omega \vec{B} \\ \vec{\nabla} \times \vec{H}_{eq} = \vec{J} + j\omega \vec{D}_{eq} \end{cases}$$

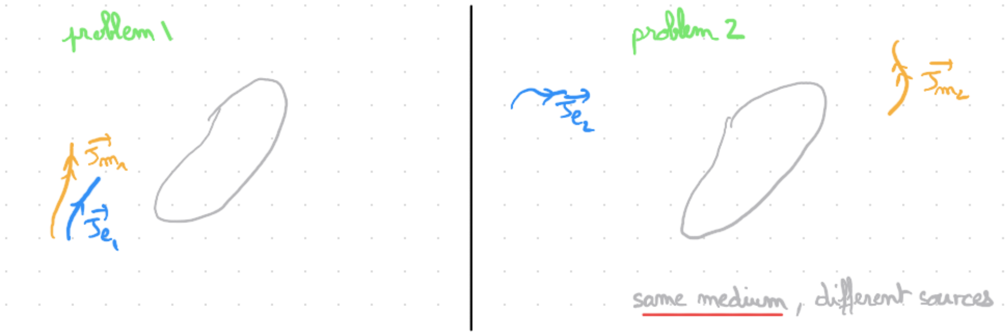
$$\vec{B} = \mu \vec{H} = \mu_{eq} \vec{H}_{eq} = \mu_{eq} \left(\vec{H} + \frac{\mu - \mu_0}{\mu_0} \vec{H}\right) = \mu_{eq} \frac{\mu}{\mu_0} \vec{H} \Rightarrow \boxed{\mu_{eq} = \mu_0}$$

$$\vec{D}_{eq} = \varepsilon \vec{E} + \frac{\mu - \mu_0}{j\omega\mu_0} \vec{\nabla} \times \underbrace{\vec{H}}_{\frac{\vec{B}}{\mu}} = \varepsilon \vec{E} + \frac{1 - \mu_r^{-1}}{j\omega\mu_0} \vec{\nabla} \times \frac{\nabla \times \vec{E}}{-j\omega}$$

$$\Rightarrow \boxed{\vec{D}_{eq} = \varepsilon \vec{E} + \varepsilon_0 \frac{1 - \mu_r^{-1}}{k_0^2} \vec{\nabla} \times \vec{\nabla} \times \vec{E}}$$

3. Reciprocity

Lorentz reciprocity theorem. (Rumsey, 1954)



The theorem relates the two problems.

$$\begin{cases} \vec{\nabla}\times\vec{E}_1 = -j\omega\vec{B}_1 - \vec{J}_{m_1} \\ \vec{\nabla}\times\vec{H}_1 = j\omega\vec{D}_1 + \vec{J}_{e_1} \end{cases} \qquad \begin{cases} \vec{\nabla}\times\vec{E}_2 = -j\omega B_2 - \vec{J}_{m_2} \\ \vec{\nabla}\times\vec{H}_2 = j\omega\vec{D}_2 + \vec{J}_{e_2} \end{cases}$$

(1) $\vec{\nabla}\cdot(\vec{E}_1\times\vec{H}_2) = \vec{H}_2\cdot(\vec{\nabla}\times\vec{E}_1) - \vec{E}_1\cdot(\vec{\nabla}\times\vec{H}_2) = -j\omega(\vec{H}_2\cdot\vec{B}_1 + \vec{E}_1\cdot\vec{D}_2) - \vec{H}_2\cdot\vec{J}_{m_1} - \vec{E}_1\cdot\vec{J}_{e_2}$

(2) $\vec{\nabla}\cdot(\vec{E}_2\times\vec{H}_1) = -j\omega(\vec{H}_1\cdot\vec{B}_2 + \vec{E}_2\cdot\vec{D}_1) - \vec{H}_1\cdot\vec{J}_{m_2} - \vec{E}_2\cdot\vec{J}_{e_1}$

(1) − (2) = $\vec{\nabla}\cdot(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = -j\omega(\vec{H}_2\cdot\vec{B}_1 - \vec{H}_1\cdot\vec{B}_2 + \vec{E}_1\cdot\vec{D}_2 - \vec{E}_2\cdot D_1)$
$$-(\vec{H}_2\cdot\vec{J}_{m_1} - \vec{H}_1\cdot\vec{J}_{m_2}) - (\vec{E}_1\cdot\vec{J}_{e_2} - \vec{E}_2\cdot\vec{J}_{e_1})$$

We integrate over a large spherical volume with $S\to S_\infty$ in vacuum:

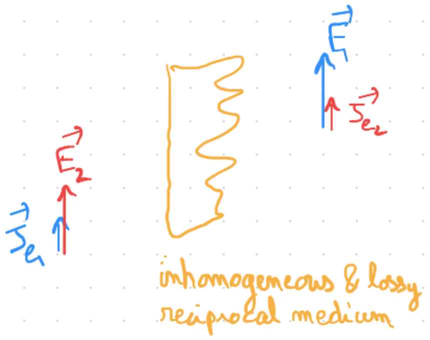
$$\int dV\, \vec{\nabla}\cdot(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) = \oint_S d\vec{S}\cdot(\vec{E}_1\times\vec{H}_2 - \vec{E}_2\times\vec{H}_1) \xrightarrow{\text{(far fields)}} 0^*$$ [[cf. proof below]]

$$\Rightarrow \underbrace{\int_V j\omega(\vec{H}_2\cdot\vec{B}_1 - \vec{H}_1\cdot\vec{B}_2 + \vec{E}_1\cdot\vec{D}_2 - \vec{E}_2\cdot\vec{D}_1)\,dV}_{\text{LHS}} = \underbrace{\int_V dV(\vec{H}_1\cdot\vec{J}_{m_2} - \vec{H}_2\cdot\vec{J}_{m_1}) + \int_V dV(\vec{E}_2\cdot\vec{J}_{e_1} - \vec{E}_1\cdot\vec{J}_{e_2})}_{\text{RHS}}$$

← vol of the sources

Lorentz reciprocity holds when the LHS vanishes so that by linearity:

RHS = 0 ⇒
$$\boxed{\begin{aligned} \int_V (\vec{J}_{e_1}\cdot\vec{E}_2)\,dV &= \int_V (\vec{J}_{e_2}\cdot\vec{E}_1)\,dV \\ \int_V (\vec{J}_{m_1}\cdot\vec{H}_2)\,dV &= \int_V (\vec{J}_{m_2}\cdot\vec{H}_1)\,dV \end{aligned}}$$
Lorentz reciprocity theorem



Application to antenna radiation pattern measurement:

[[* In far field, $\vec{E}_1 = \eta_0(\vec{H}_1 \times \vec{e_r})$, $\vec{E}_2 = \eta_0(\vec{H}_2 \times \vec{e_r})$ with $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$. (quasi plane waves)

$$\Rightarrow \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1 = \eta_0 \left\{ \vec{H}_1 \times \vec{e_r} \times \vec{H}_2 - \vec{H}_2 \times \vec{e_r} \times \vec{H}_1 \right\} =$$

$$\eta_0 \left\{ \vec{e_r}\cancel{(\vec{H}_1 \cdot \vec{H}_2)} - \vec{H}_2 \underbrace{(\vec{H}_1 \cdot \vec{e_r})}_{\to 0} - \vec{e_r}\cancel{(\vec{H}_1 \cdot \vec{H}_2)} + \vec{H}_1 \underbrace{(\vec{H}_2 \cdot \vec{e_r})}_{\to 0} \right\} \xrightarrow{\text{far field}} 0$$]]

Conditions for reciprocity to hold

isotropic case

$$\text{LHS} = \vec{H}_2 \cdot \mu \vec{H}_1 - \vec{H}_1 \cdot \mu \vec{H}_2 + \vec{E}_1 \cdot \varepsilon \vec{E}_2 - \vec{E}_2 \cdot \varepsilon \vec{E}_1 = 0$$ always reciprocal

even if $\begin{cases} \varepsilon = \varepsilon_R(\vec{r}) + j\,\varepsilon_I(\vec{r}) \\ \mu = \mu_R(\vec{r}) + j\mu_I(\vec{r}) \end{cases}$ lossy and inhomogeneous medium !

$\downarrow$

quite surprising

bi-anisotropic case

$$\text{LHS} = 0 \Leftrightarrow \vec{H}_2 \cdot \hat{\mu} \vec{H}_1 + \vec{H}_2 \cdot \hat{\beta} \vec{E}_1 - \vec{H}_1 \cdot \hat{\mu} \vec{H}_2 - \vec{H}_1 \cdot \hat{\beta} \vec{E}_2 + \vec{E}_1 \cdot \hat{\varepsilon} \vec{E}_2 + \vec{E}_1 \cdot \hat{\alpha} \vec{H}_2 - \vec{E}_2 \cdot \hat{\varepsilon} \vec{E}_1 - \vec{E}_2 \hat{\alpha} \vec{E}_1 = 0$$

$$\vec{A} \cdot m \vec{B} = A_i\, m_{ij}\, B_j = A_j\, m_{ji}\, B_i = \vec{A} \cdot m^T \vec{B}$$

$$\Rightarrow \boxed{\begin{aligned} * \ \hat{\mu} &= \hat{\mu}^T \\ * \ \hat{\varepsilon} &= \hat{\varepsilon}^T \\ * \ \hat{\beta} &= -\hat{\alpha}^T \end{aligned}}$$

conditions for reciprocal bi-anisotropic media

Reminder :

$$\boxed{\begin{aligned} \hat{\mu} &= \hat{\mu}^\dagger \\ \hat{\varepsilon} &= \hat{\varepsilon}^\dagger \\ \hat{\alpha} &= \beta^\dagger \end{aligned}}$$

condition for lossless bi-anisotropic media

Rq: The reciprocal conditions are an example of Onsager-Casimir relations of microscopic reversibility :

$$\boxed{\alpha_{ik}(\omega, \vec{H}_0) = \pm\ \alpha_{ik}(\omega, -\vec{H}_0)}$$

Onsager's reciprocity relations for linear time invariant sys.

with α_{ik} = generalized susceptibilities, $\vec{H}_0$ a set of time-odd external parameters ($\vec{H}_0 = 0$ is required for reciprocity to hold) and $\pm$ when α_{ik} connects quantities with same (+) or different (-) parities with respect to time-reversal.

ex: Lossless reciprocal chiral media :

$$\begin{cases} \vec{D} = \varepsilon \vec{E} + \alpha \vec{H} \\ \vec{B} = \mu \vec{H} + \beta \vec{E} \end{cases} \Rightarrow \begin{cases} \varepsilon, \mu \in \mathbb{R} \\ \beta = -\alpha = \alpha^* \Rightarrow \alpha \in j\mathbb{R},\ \alpha = -\beta \end{cases}$$

$$\boxed{\begin{aligned} \vec{D} &= \varepsilon \vec{E} + j\gamma \vec{H} \\ \vec{B} &= \mu \vec{H} - j\gamma \vec{E} \end{aligned}}$$

lossless reciprocal chiral